Cut Finite Element Approximation of the Mean Curvature Vector

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Overview

Introduction

Motivation

Mechanics

- Multiphase fluid flows
- Gurtin-Murdoch theory of deformable surfaces and nanoelasticity
- Deformation of soft materials

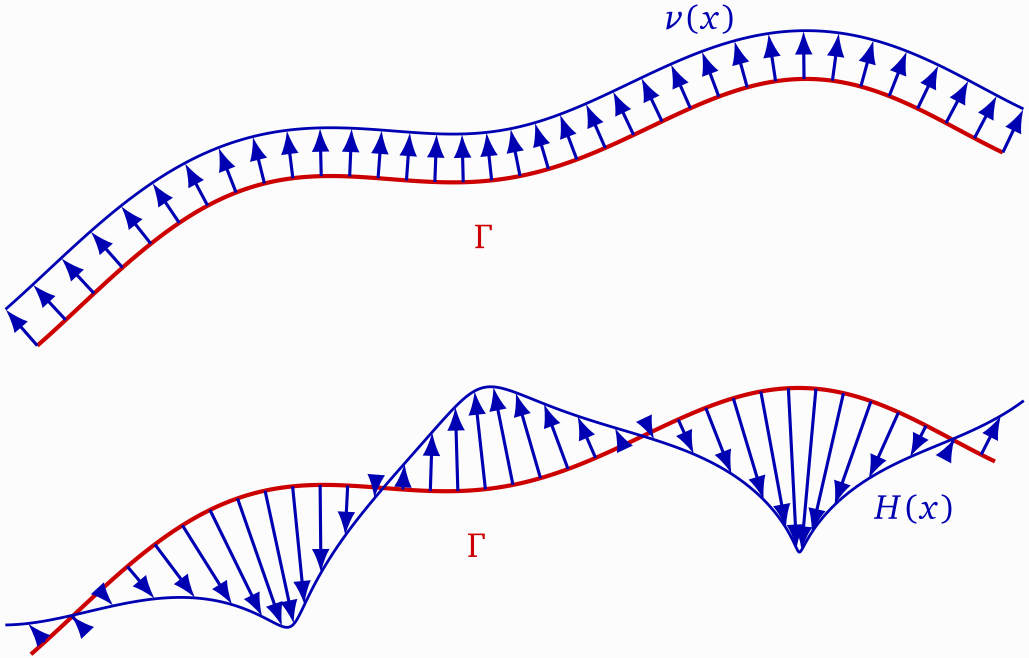
Materials science

- Dendritic crystal growth and solidification
- Phase separation dynamics on manifolds

Mathematical Preliminaries

Mean Curvature

- Let $\Gamma \subset \mathbb{R}^d$ be an orientable C^2 surface with outward unit normal $\nu : \Gamma \to \mathbb{R}^d$
- The mean curvature can be computed as $\kappa(x) = \nabla_\Gamma \cdot \nu(x)$
- The mean curvature vector is defined by $H(x) = \kappa(x)\nu(x)$
- The mean curvature vector can also be computed using the Laplace–Beltrami operator $H = -\Delta_\Gamma x_\Gamma$



Mean Curvature – Weak Problem

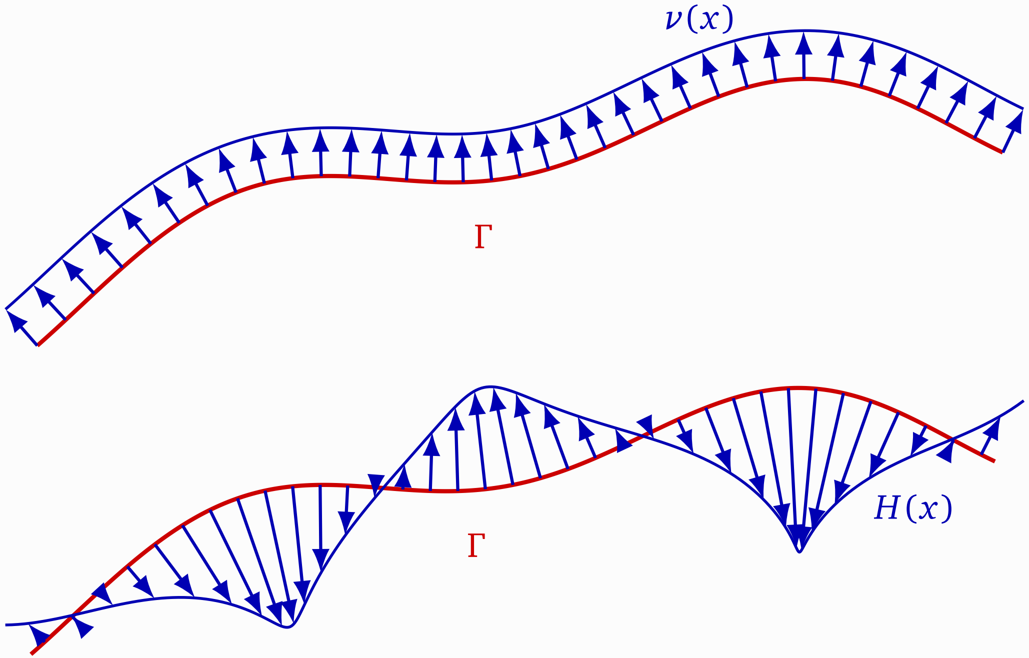
Strong problem:

$$\begin{cases} \text{Find } H \in L^2(\Gamma, \mathbb{R}^d) & \text{such that} \\ H = -\Delta_\Gamma x_\Gamma. \end{cases}$$

1. Let $v \in H^1(\Gamma, \mathbb{R}^d)$
2. $\int_\Gamma H \cdot v \, d\Gamma = -\int_\Gamma \Delta_\Gamma x_\Gamma \cdot v \, d\Gamma$
3. $\underbrace{\int_\Gamma H \cdot v \, d\Gamma}_{a(H,v)} = \underbrace{\int_\Gamma \nabla_\Gamma x_\Gamma \cdot \nabla_\Gamma v \, d\Gamma}_{L(v)}$

Weak problem:

$$\begin{cases} \text{Find } H \in L^2(\Gamma, \mathbb{R}^d) & \text{such that} \\ a(H, v) = L(v), & \forall v \in H^1(\Gamma, \mathbb{R}^d). \end{cases}$$



Finite Element Approximation

Continuous weak problem:

$$\begin{cases} \text{Find } H \in L^2(\Gamma, \mathbb{R}^d) \text{ such that} \\ a(H, v) = L(v), \quad \forall v \in H^1(\Gamma, \mathbb{R}^d). \end{cases}$$

Replace by a finite-dimensional problem on an approximate surface:

$$\Gamma \rightsquigarrow \Gamma_h, \qquad H^1(\Gamma, \mathbb{R}^d) \rightsquigarrow V_h.$$

Usually V_h is a space of continuous piecewise polynomial functions.

Discrete weak problem:

$$\begin{cases} \text{Find } H_h \in V_h \text{ such that} \\ a_h(H_h, v) = L_h(v), \quad \forall v \in V_h. \end{cases}$$

What makes this delicate?

Curvature depends on derivatives of the geometry.

$$\text{surface} \longrightarrow \text{normal} \longrightarrow \text{curvature}$$

Rule of thumb

Each derivative typically costs one order of accuracy.

$$\text{degree } k \quad \rightsquigarrow \quad \text{accuracy } k-1.$$

Cut Finite Element Methods

Why CutFEM?

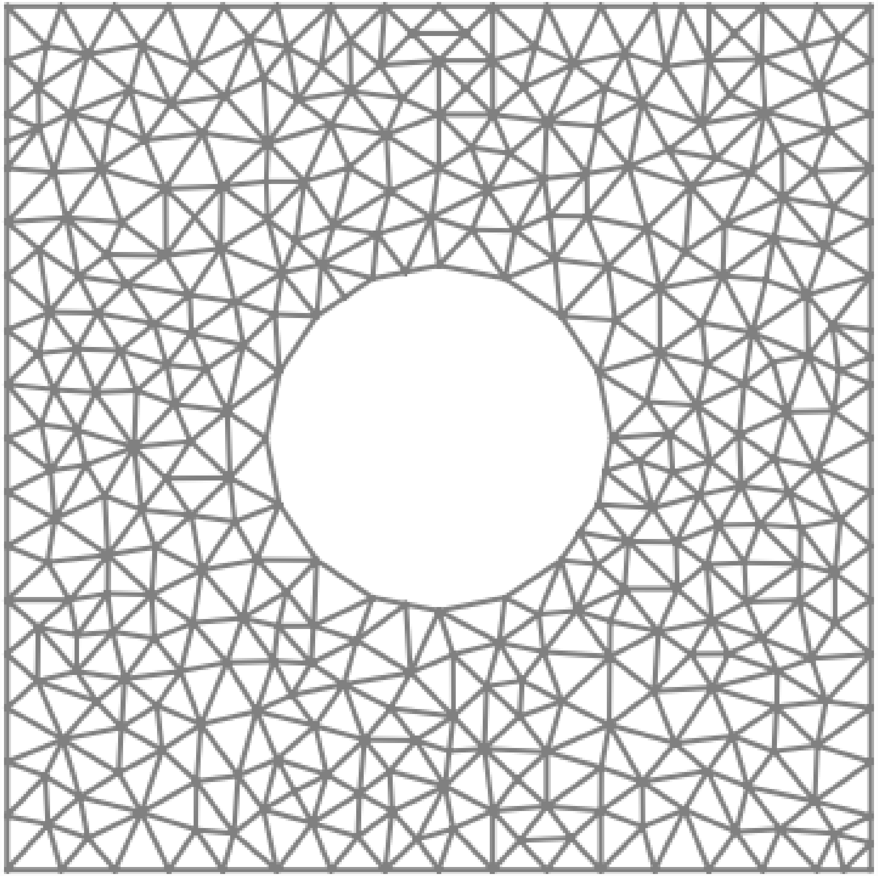
Body-fitted FEM

The mesh follows the moving domain.

- Natural for fixed domains
- For evolving domains, the mesh must deform
- At some point, re-meshing may be needed

CutFEM idea

Keep the background mesh fixed and let the geometry cut through it.



Linear CutFEM for Mean Curvature

Originally formulated in 2015 by Hansbo, Larson, & Zahedi.

Let Γ_h be a family of piecewise planar approximations of Γ.

$$\|x - p(x)\|_{L^\infty(\Gamma_h)} \lesssim h^2,$$

$$\|\nu \circ p - \nu_h\|_{L^\infty(\Gamma_h)} \lesssim h.$$

Active mesh:

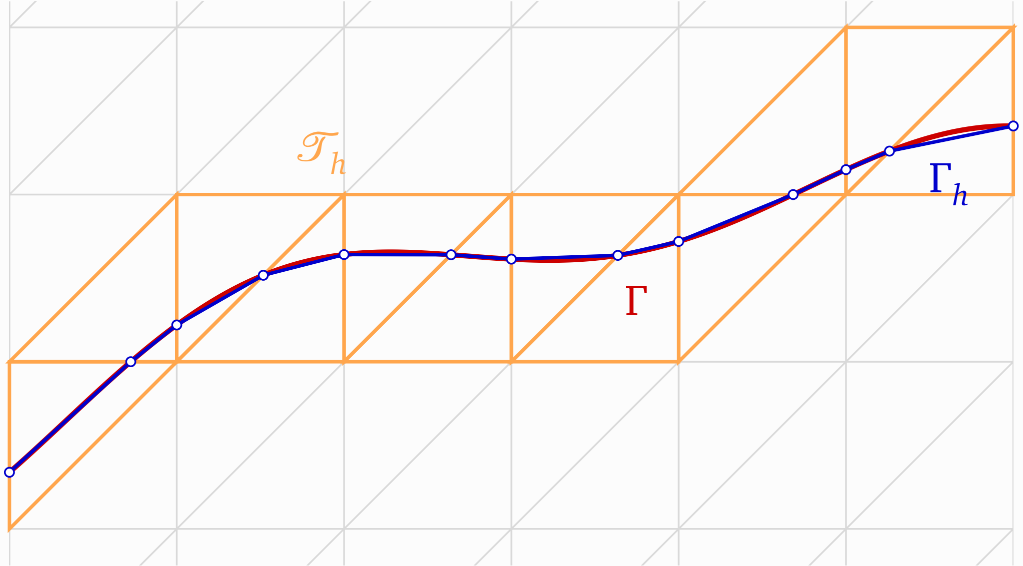
$$\mathcal{T}_h = \{T \in \mathcal{T}_{h,0} : T \cap \Gamma_h \neq \emptyset\}.$$

Surface partition:

$$\mathcal{K}_h = \{T \cap \Gamma_h : T \in \mathcal{T}_h\}.$$

Finite element space:

$$V_h = \{v \in C^0(\Omega_h) : v|_T \in \mathbb{P}_1(T)\}.$$



Linear CutFEM for Mean Curvature

Discrete weak problem:

$$\begin{cases} \text{Find } H_h \in [V_h]^d \text{ such that} \\ A_h(H_h, v) = L_h(v), \quad \forall v \in [V_h]^d, \end{cases}$$

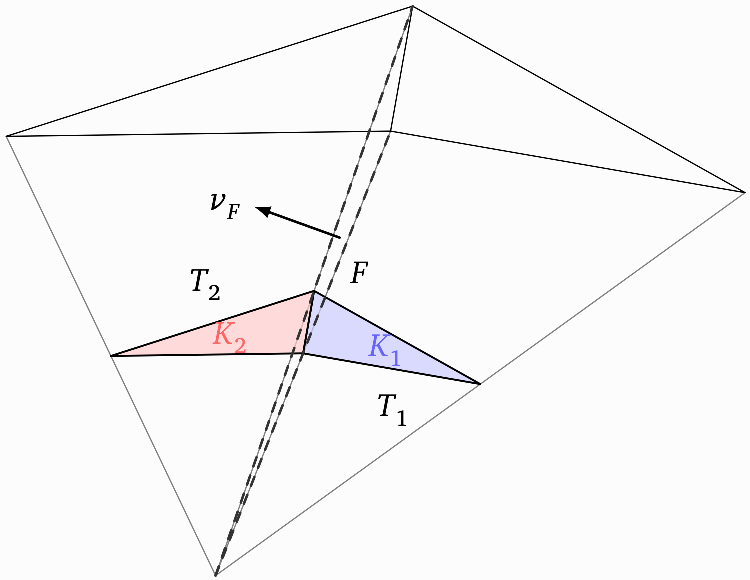
With

$$A_h(u,v) = a_h(u,v) + s_{h,F}(u,v)$$

$$a_h(u,v) = \langle u, v \rangle_{\Gamma_h} := \int_{\Gamma_h} u \cdot v \, d\Gamma_h$$

$$L_h(v) = \left\langle \nabla_{\Gamma_h} x_{\Gamma_h}, \nabla_{\Gamma_h} v \right\rangle_{\Gamma_h}$$

$$s_{h,F}(u,v) = \sum_{F \in \mathscr{F}_h} \tau_F \langle [\nu_F \cdot \nabla u], [\nu_F \cdot \nabla v] \rangle_F .$$



$$[\nu_F \cdot \nabla u] = \nu_F \cdot \nabla u_1 - \nu_F \cdot \nabla u_2$$

What Must the Stabilization Do?

The first contribution of this thesis is to abstract the properties the stabilization form. We assume that s_h satisfies

(A1) symmetric semi-positive definiteness, defining a semi-norm $\|\cdot\|_{s_h}$

(A2) weak consistency:

$$\|H^e\|_{s_h} + \|\pi_h H^e\|_{s_h} \lesssim h$$

(A3) tangential-gradient and trace control:

$$h^2 \left\|\nabla_{\Gamma_h} v\right\|^2_{L^2(\Gamma_h)} + \sum_{K \in \mathcal{K}_h} h \|v\|^2_{L^2(\partial K)} + \sum_{E \in \mathcal{E}_h} h \left\|[t_E \cdot \nabla_{\Gamma_h} v]\right\|^2_{L^2(E)} \lesssim \|v\|^2_{L^2(\Gamma_h)} + \|v\|^2_{s_h} .$$

Error Bound – Linear Case

Theorem

If the geometric assumptions hold, and the stabilization form s_h satisfies (A1)–(A3), the solution H_h to the discrete problem satisfies the following bound[1]:

$$\|H^e - H_h\|^2_{L^2(\Gamma_h)} + \|H^e - H_h\|^2_{s_h} \lesssim h^2.$$

[1] *Stabilized finite element approximation of the mean curvature vector on closed surfaces*, P. Hansbo, M. G. Larson & S. Zahedi, 2015

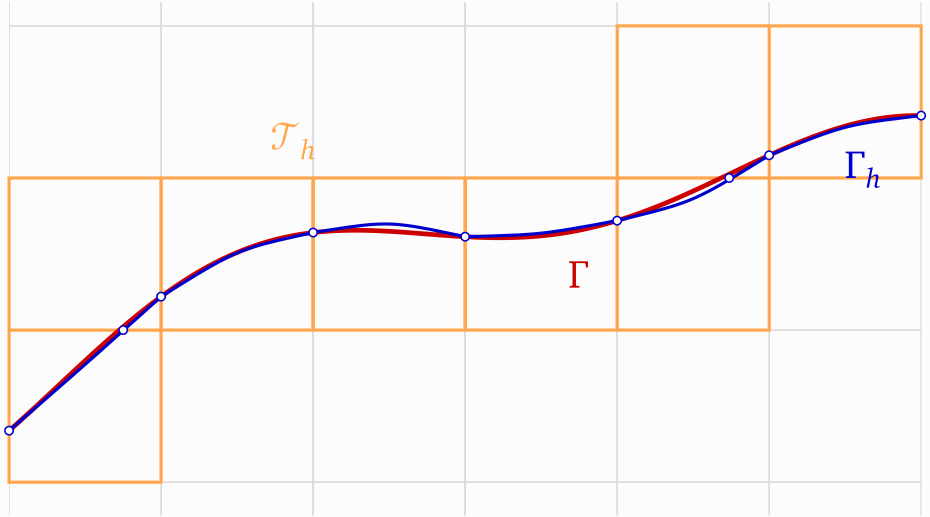
Extension to $\mathbb{Q}_1$ Elements

The second contribution is to extend the framework to tensor-product elements

- Replace simplices by orthogonal hyper-rectangles:

  $$V_h = \{v \in C^0(\Omega_h) : v|_T \in \mathbb{Q}_1(T)\}.$$

  $\mathbb{Q}_1$ has basis functions $\{1, x, y, xy\}$ in $\mathbb{R}^2$
- $\Gamma_h \cap T$ can now be curved, but the approximation bounds stay the same



Extension to $\mathbb{Q}_1$ Elements

- The error bound proof for the linear case relies on the fact that the surface Laplacian disappears for linear function on flat surfaces

$$\mathbb{P}_1 : \Delta_{\Gamma_h} v = 0 \quad \forall v \in V_h$$

- This is not the case for $\mathbb{Q}_1$ functions or curved discrete surfaces

$$\mathbb{Q}_1 : \Delta_{\Gamma_h} v \neq 0 \quad \forall v \in V_h$$

- By introducing an additional assumption on the stabilization form we are able to recover the first order error estimate

(A4) We assume s_h satisfies:

$$h^2 \|\Delta_{\Gamma_h} v\|^2_{L^2(\Gamma_h)} \lesssim \|v\|^2_{L^2(\Gamma_h)} + \|v\|^2_{s_h}$$

Error Bound – Multilinear Case

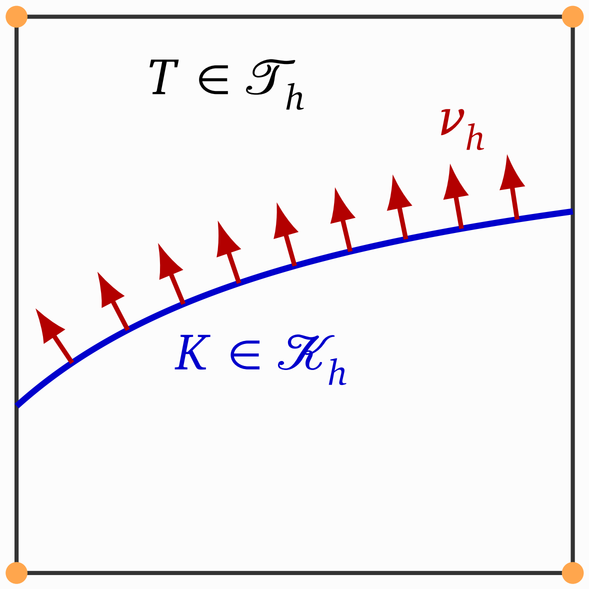
Theorem

If the geometric assumptions hold, and the stabilization form s_h satisfies (A1)–(A3), and (A4), the solution H_h to the discrete problem satisfies the following bound:

$$\|H^e - H_h\|^2_{L^2(\Gamma_h)} + \|H^e - H_h\|^2_{s_h} \lesssim h^2.$$

Proposition

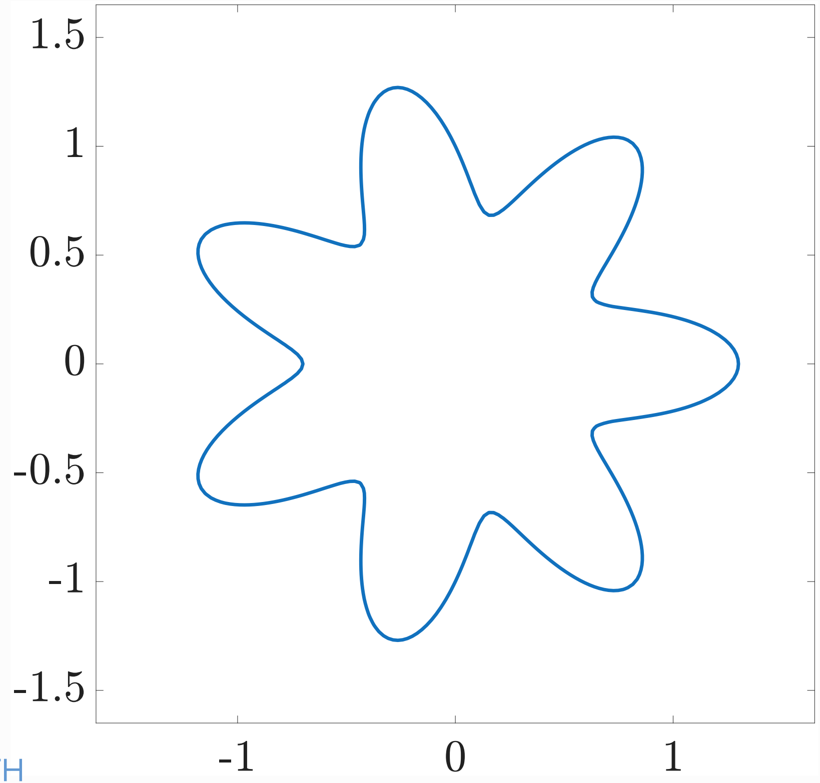
The stabilization form $s_h = s_{h,F} + s_{h,N}$ satisfies assumptions (A1)–(A4).



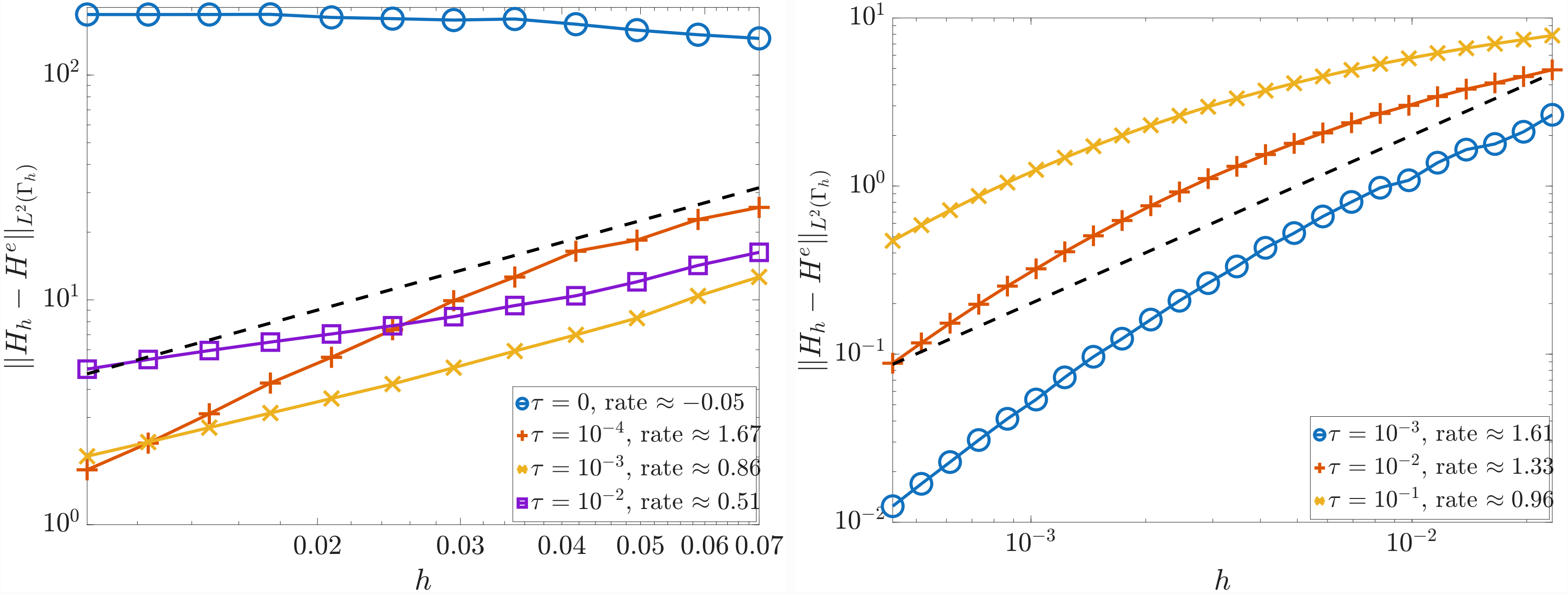
$$s_{h,N}(u,v) = \sum_{T \in \mathscr{T}_h} \tau_N \langle \nu_h \cdot \nabla u, \nu_h \cdot \nabla v \rangle_T$$

Numerical Examples

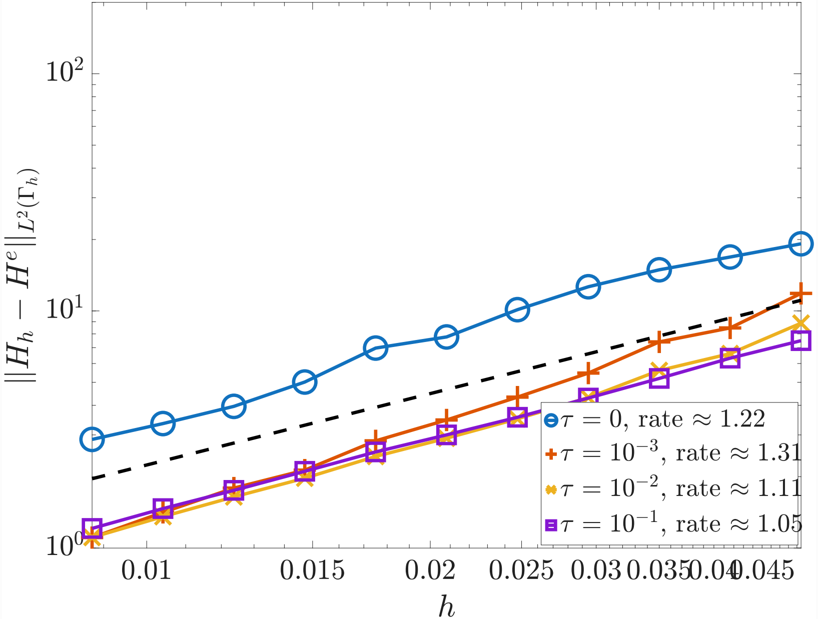
Example geometries

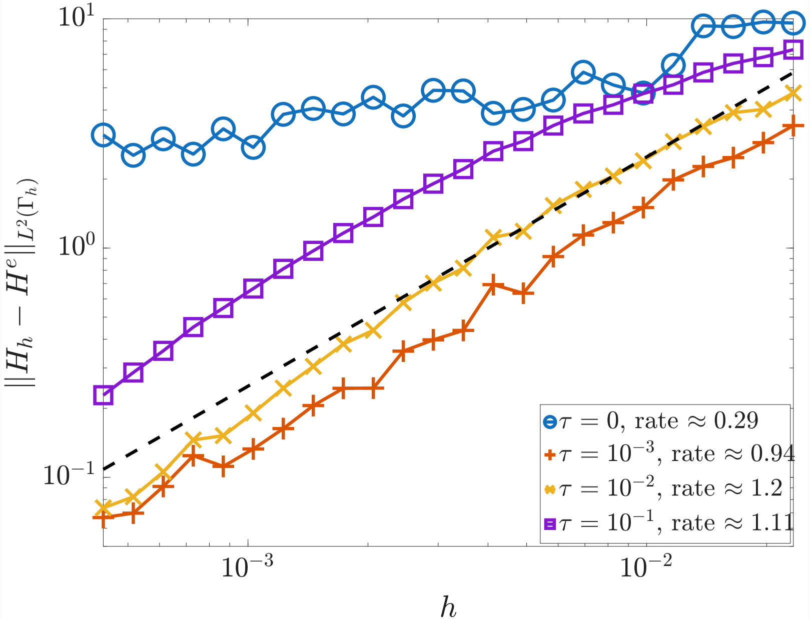


$\mathbb{P}_1$ examples



$\mathbb{Q}_1$ examples





Conclusion & Outlook

- The linear CutFEM theory can be formulated using abstract assumptions on s_h.
- For $\mathbb{Q}_1$ finite elements, it is possible to restore the first order bound.
- Numerical experiments agree with the theory and show robustness to parameter choice and mesh location.

Outlook

In the thesis report we also look at the conditioning of the linear system of equations and higher-order polynomial spaces are treated, but closing an order k bound remains an open problem.

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