Definitions used in Cohn's *Measure Theory* (second edition)

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1 Measures

Definition 1.1 (Algebra)**.** Let X be a set, an arbitrary collection of subsets $\mathscr{A}$ of X is an *algebra* on X if

(a) $X \in \mathscr{A}$,

(b) if $A \in \mathscr{A}$ then $A^c \in \mathscr{A}$,

(c) for each finite sequence $\{A_n\}_{n=1}^N$ of sets in $\mathscr{A}$, the set $\bigcup_{n=1}^N A_n$ belongs to $\mathscr{A}$, and

(d) for each finite sequence $\{A_n\}_{n=1}^N$ of sets in $\mathscr{A}$, the set $\bigcap_{n=1}^N A_n$ belongs to $\mathscr{A}$.

Definition 1.2 (σ-Algebra)**.** Let X be a set, an arbitrary collection of subsets $\mathscr{A}$ of X is a σ-*algebra* on X if

(a) $X \in \mathscr{A}$,

(b) if $A \in \mathscr{A}$ then $A^c \in \mathscr{A}$,

(c) for each infinite sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathscr{A}$, the set $\bigcup_{n=1}^\infty A_n$ belongs to $\mathscr{A}$, and

(d) for each infinite sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathscr{A}$, the set $\bigcap_{n=1}^\infty A_n$ belongs to $\mathscr{A}$.

Definition 1.3 (Borel σ-algebra on $\mathbb{R}^d$)**.** The *Borel σ-algebra on* $\mathbb{R}^d$, denoted $\mathscr{B}(\mathbb{R}^d)$, is generated by the collection of open subsets of $\mathbb{R}^d$. **Proposition 1.1.5** states that $\mathscr{B}(\mathbb{R}^d)$ is generated by each of the collections of sets

(a) the collection of all closed subsets of $\mathbb{R}^d$;

(b) the collection of all closed half-spaces in $\mathbb{R}^d$ that have the form $\{(x_1, \dots, x_d) : x_i \le b\}$ for some $b \in \mathbb{R}$;

(c) the collection of all rectangles in $\mathbb{R}^d$ that have the form

$$\big\{(x_1, \dots, x_d) : a_i < x_i \le b_i \text{ for } i = 1, \dots, d\big\}.$$

Definition 1.4 (Measure)**.** Let $\mathscr{A}$ be a σ-algebra. A function $\mu : \mathscr{A} \to [0, \infty]$ is called *countably additive* if

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n),$$

for each infinite sequence of disjoint sets $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{A}$. If μ in addition to being countably additive also satisfies $\mu(\varnothing) = 0$, μ is said to be a *measure* on $\mathscr{A}$.

Definition 1.5 (Measure space)**.** Let X be a set, $\mathscr{A}$ a σ-algebra on X and μ a measure on $\mathscr{A}$. The triplet $(X, \mathscr{A}, \mu)$ is then called a *measure space*, the pair $(X, \mathscr{A})$ is often called a *measurable space.*

Definition 1.6 (Outer measure)**.** Let X be a set, and let $\mathscr{P}(X)$ be the power set of X. An *outer measure* on X is a function $\mu^* : \mathscr{P}(X) \to [0, \infty]$ such that

(a) $\mu^*(\varnothing) = 0$,

(b) $A \subseteq B \subseteq X$ implies $\mu^*(A) \leq \mu^*(B)$, and

(c) if $\{A_n\}$ is an infinite sequence of sets in $\mathscr{P}(X)$, then $\mu^*\left(\bigcup A_n\right) \leq \sum \mu^*(A_n)$.

Definition 1.7 (Lebesgue outer measure)**.** *Lebesgue outer measure* on $\mathbb{R}^d$ which we denote by λ^* is defined as follows. For each set $A \subseteq \mathbb{R}^d$ define the set $\mathscr{C}_A$ of all sequences $\{R_n\}$ of bounded and open d-cells R_n such that $A \subseteq \bigcup_{n=1}^{\infty} R_n$. Then

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{vol}(R_n) : \{R_n\} \in \mathscr{C}_A \right\}.$$

Definition 1.8 (μ^*-measurable set)**.** Let X be a set, and let μ^* be an outer measure on X. A subset B of X is *μ^*-measurable* if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c),$$

for all $A \subseteq X$.

Definition 1.9 (Complete measure)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. The measure μ, or the measure space $(X, \mathscr{A}, \mu)$, is called *complete* if $A \in \mathscr{A}$, $\mu(A) = 0$, and $B \subseteq A$ implies $B \in \mathscr{A}$.

Definition 1.10 (μ-negligible set)**.** A subset B of X is called *μ-negligible* or *μ-null* if there exists $A \in \mathscr{A}$ such that $\mu(A) = 0$ and $B \subseteq A$. Thus $(X, \mathscr{A}, \mu)$ is complete if and only if every μ-negligible set belongs to $\mathscr{A}$.

Definition 1.11 (Completion of a σ-algebra under a measure)**.** Let $(X, \mathscr{A})$ be a measurable space. The *completion* of $\mathscr{A}$ under μ is the collection $\mathscr{A}_\mu$ of $A \subseteq X$ for which there exists $E, F \in \mathscr{A}$ such that

$$E \subseteq A \subseteq F,$$

and

$$\mu(F \setminus E) = 0.$$

A set that belongs to $\mathscr{A}_\mu$ is sometimes said to be *μ-measurable*.

Definition 1.12 (Completion of a measure)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. The *completion* of μ is defined as $\overline{\mu} : \mathscr{A}_\mu \to [0, \infty]$ by letting $\overline{\mu}(A)$ be the common value of E, F, defined in the above definition.

Definition 1.13 (Inner and outer measure)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let A be an arbitrary subset of X. The *inner measure* μ_* of A is defined by

$$\mu_*(A) = \sup \{\mu(B) : B \subseteq A \text{ and } B \in \mathscr{A}\} .$$

The *outer measure* μ^* of A meanwhile, is defined by

$$\mu^*(A) = \inf \{\mu(B) : A \subseteq B \text{ and } B \in \mathscr{A}\} .$$

Remark. According to **Proposition 1.5.4**, the outer measure defined in **Definition 1.13** satisfies the conditions placed on an outer measure in **Definition 1.6**.

Definition 1.14 (Regular measure)**.** Let $\mathscr{A}$ be a σ-algebra on $\mathbb{R}^d$ that includes $\mathscr{B}(\mathbb{R}^d)$. A measure μ on $\mathscr{A}$ is regular if

(a) each compact subset K of $\mathbb{R}^d$ satisfies $\mu(K) < \infty$,

(b) each set A in $\mathscr{A}$ satisfies

$$\mu(A) = \inf \{\mu(U) : U \text{ is open and} A \subseteq U\} , \textit{and}$$

(c) each open subset U of $\mathbb{R}^d$ satisfies

$$\mu(U) = \sup \{\mu(K) : K \text{ is compact and} K \subseteq A\} .$$

Definition 1.15 (Dykin class)**.** Let X be a set. A collection $\mathscr{D}$ is a *d-system*, or *Dykin class*, on X if

(a) $X \in \mathscr{D}$,

(b) $A \setminus B \in \mathscr{D}$ whenever $A, B \in \mathscr{D}$ and $A \supseteq B$, and

(c) $\bigcup A_n \in \mathscr{D}$ whenever $\{A_n\}$ is an increasing sequence of sets in $\mathscr{D}$.

Definition 1.16 (π-system)**.** A collection of subsets of X is a *π-system* if it is closed under the formation of finite unions.

2 Functions and Integrals

Definition 2.1 ($\mathscr{A}$-measurable function)**.** Let $(X, \mathscr{A})$ be a measurable space, and let $A \in \mathscr{A}$. A function $f : A \to [-\infty, \infty]$ is *measurable with respect to* $\mathscr{A}$ if it satisfies any of the conditions, and thus all, of the conditions in **Proposition 2.1.1**. That is any of

(a) $\forall t \in \mathbb{R} \quad \{x \in A : f(x) \leq t\} \in \mathscr{A}$,

(b) $\forall t \in \mathbb{R} \quad \{x \in A : f(x) < t\} \in \mathscr{A}$,

(c) $\forall t \in \mathbb{R} \quad \{x \in A : f(x) \geq t\} \in \mathscr{A}$,

(d) $\forall t \in \mathbb{R} \quad \{x \in A : f(x) > t\} \in \mathscr{A}$.

A function that is measurable with respect to $\mathscr{A}$ may be called *$\mathscr{A}$-measurable* or if what σ-algebra is meant is obvious from context, simply *measurable*. In the case $X = \mathbb{R}^d$ functions measurable with respect to $\mathscr{B}(\mathbb{R}^d)$ are called *Borel measurable* or *Borel functions*. A function measurable with respect to $\mathscr{M}_{\lambda^*}$ is called *Lebesgue measurable*.

Definition 2.2 (Almost everywhere)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. A property of points on X is said to hold *μ-almost everywhere* if the set of points in X where it fails to hold is μ-negligible. The expression μ-almost everywhere is often abbreviated μ-a.e. or to a.e.[μ]. If the measure is clear from context one may simply say *almost everywhere*.

2.1 Construction of the integral

Definition 2.3 (Integral of a simple non-negative function)**.** Let μ be a measure on $(X, \mathscr{A})$. If f is a real-valued, simple, $\mathscr{A}$-measurable function given by $f = \sum_{i=1}^{m} a_i \chi_{A_i}$, where each $a_i \geq 0$ and $A_i \in \mathscr{A}$ are disjoint. Then the *integral of f with respect to μ* is then defined to be

$$\int f \, d\mu = \sum_{n=1}^{m} a_i \mu(A_i).$$

Definition 2.4 (Integral of arbitrary $\mathscr{A}$-measurable, non-negative function)**.** Let f be an arbitrary $\mathscr{A}$-measurable function, with image in $[0, \infty]$. The integral of f is then defined as

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \in \mathscr{S}_+ \text{ and } g \leq f \right\}.$$

Definition 2.5 (Integral of arbitrary measurable function)**.** Let $f : X \to [-\infty, \infty]$ be a measurable function on $(X, \mathscr{A}, \mu)$. If $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are both finite, then f is called *integrable* and its *integral* is defined by

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

The integral of f is said to *exist* if at least one of $\int f^+\,d\mu$ and $\int f^-\,d\mu$ is finite, in this case the integral is defined $\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu$.

Definition 2.6 (Integral over a subset)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $f : X \to [-\infty, \infty]$ be $\mathscr{A}$-measurable. The integral of f over a subset $A \subseteq X$ is said to exist if the integral of $f\chi_A$ exists. In that case the integral over A is defined to be

$$\int_A f\,d\mu = \int f\chi_A\,d\mu.$$

Likewise, if $A \in \mathscr{A}$ and $f : A \to [-\infty, \infty]$ is $\mathscr{A}$-measurable, then the integral of f over A is defined to be the integral of the function which agrees with f on A and vanishes on A^c.

Definition 2.7 (Lebesgue integral)**.** The case $X = \mathbb{R}^d$ and $\mu = \lambda$ we simply talk about *Lebesgue integrability* and the *Lebesgue integral*. We may use any of the following notations for the *Lebesgue integral* over an interval $[a, b]$

$$\int_a^b f = \int_a^b f(x)\,dx = (L)\int_a^b f = (L)\int_a^b f(x)\,dx,$$

where the latter two are used to emphasise that we are talking about the Lebesgue integral.

Definition 2.8 ($\mathscr{L}^1$)**.** We define $\mathscr{L}^1(X, \mathscr{A}, \mu, \mathbb{R})$, or sometimes simply $\mathscr{L}^1$, as the set of all integrable functions $f : X \to \mathbb{R}$. (As opposed to $[-\infty, \infty]$-valued functions.)

2.2 Measurable functions again

Definition 2.9 (Measurable function between sets)**.** Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A function $f : X \to Y$ is *measurable with respect to $\mathscr{A}$ and $\mathscr{B}$* if for each $B \in \mathscr{B}$ the set $f^{-1}(B)$ belongs to $\mathscr{A}$. In stead of saying measurable with respect to $\mathscr{A}$ and $\mathscr{B}$, we may say that f is a *measurable function* from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$, or simply that $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is *measurable*.

Definition 2.10 (Integral of complex-valued function)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. A complex-valued function f on X is *integrable* if its real and imaginary parts $\Re(f)$ and $\Im(f)$ are integrable; if f is integrable then its *integral* is defined by

$$\int f\,d\mu = \int \Re(f)\,d\mu + i\int \Im(f)\,d\mu.$$

Definition 2.11 (μf^{-1})**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, let $(Y, \mathscr{B})$ be a measurable space, and let $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be measurable.Define $\mu f^{-1} : \mathscr{B} \to [0, \infty]$ by $\mu f^{-1}(B) = \mu(f^{-1}(B))$. It is easy to show that μf^{-1} is a measure, this measure is sometimes called the *image of μ under f*.

3 Convergence

Definition 3.1 (Convergence in measure)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \dots$ be real valued $\mathscr{A}$-measurable functions on X. The sequence $\{f_n\}$ converges to f *in measure* if

$$\lim_n \mu\left(\{x \in X : |f_n(x) - f_n| > \varepsilon\}\right) = 0$$

for every $\varepsilon > 0$.

Definition 3.2 (Almost uniform convergence)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \dots$ be real valued $\mathscr{A}$-measurable functions on X. Then $\{f_n\}$ converges to f *almost uniformly* if for all $\varepsilon > 0$ there is $B \in \mathscr{A}$ such that $\{f_n\}$ converges to f on B and $\mu(B^c) < \varepsilon$.

Definition 3.3 (Convergence in mean)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \dots$ be real valued $\mathscr{A}$-measurable functions on X. Then $\{f_n\}$ converges to f *in mean* if

$$\lim_n \int |f_n - f| \, d\mu = 0.$$

3.1 Normed spaces

Definition 3.4 (Norm & seminorm)**.** Let V be a vector space over $\mathbb{C}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies

(a) $\|v\| \geq 0$,

(b) $\|v\| = 0 \iff v = 0$,

(c) $\|\alpha v\| = |\alpha| \, \|v\|$,

(d) $\|u + v\| \leq \|u\| + \|v\|$

for each $u, v \in V$ and $\alpha \in \mathbb{C}$. If condition (b) was replaced by "$\|v\| = 0 \Longleftarrow v = 0$" $\|\cdot\|$ is a *seminorm.*

Definition 3.5 (Metric & semimetric)**.** A *metric* on a set S is a function $d : S \times S \to \mathbb{R}$ that satisfies

(a) $d(s,t) \geq 0$,

(b) $d(s,t) = 0 \iff s = t$,

(c) $d(s,t) = d(t,s)$,

(d) $d(r,t) \leq d(r,s) + d(s,t)$

for all $r, s, t \in S$. If condition (b) is replaced by "$d(s,t) = 0 \Longleftarrow s = t$" d is a *semimetric.* A *metric space* is a set S together with a metric d on S. This may, if there is no risk for confusion with a measurable space, be written as (S, d).

Definition 3.6 (Converging sequence)**.** Let (S, d) be a metric (or semimetric) space, a sequence $\{s_n\}$ in S is said to *converge* to $s \in S$ if for all $\varepsilon > 0$ there exists N such that $\forall n \geq N$ $d(s_n, s) \leq \varepsilon$. The point s is then said to be the *limit point* of $\{s_n\}$. In particular, if V is a normed linear space, $v \in V$ and $\{v_n\}$ is a sequence in V, then $\{v_n\}$ converges to v (with respect to the metric induced by the norm on V) if and only if $\lim_n \|v_n - v\| = 0$. Note that if d is a semimetric $\{s_n\}$ may have several limit points.

Definition 3.7 (Dense subset)**.** Let (S, d) be a metric (or semimetric) space, a subset $A \subseteq S$ is said to be *dense* in S if for all $s \in S$ and $\varepsilon > 0$ there exists $a \in A$ such that $d(s, a) < \varepsilon$.

Definition 3.8 (Separable space)**.** Let (S, d) be a metric (or semimetric) space, if S has a countable dense subset, S is *separable.*

Definition 3.9 (Cauchy sequences and completeness)**.** Let (S, d) be a metric space, a *Cauchy sequence* is a sequence $\{s_n\}$ in S such that for all $\varepsilon > 0$ there exists N such that for all $n, m \geq N$, $d(s_n, s_m) < \varepsilon$. A metric space (S, d) is said to be *complete* if all Cauchy sequences in (S, d) converge.

Definition 3.10 (Banach space)**.** If a normed linear space is complete, with respect to the metric induced by the norm on the space, then it is called a *Banach space.*

Definition 3.11 (Inner product)**.** Let V be a vector space over $\mathbb{C}$. A function $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is an *inner product* on V if

(a) $(x, x) \geq 0$,

(b) $(x, x) = 0 \Longleftrightarrow x = 0$,

(c) $(x, y) = \overline{(y, x)}$, and

(d) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

hold for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$. An *inner product space* is a vector space, together with an inner product. The *norm* $\|\cdot\|$ associated to the inner product $(\cdot, \cdot)$ is defined by $\|x\| = \sqrt{(x, x)}$.

Definition 3.12 (Hilbert space)**.** An inner product space that is complete under the norm associated with the inner product is called a *Hilbert space.*

3.2 $\mathscr{L}^p$ and L^p

Definition 3.13 ($\mathscr{L}^p$)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $p \in [1, \infty)$. Then $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{R})$ is the set of all $\mathscr{A}$-measurable functions $f : X \to \mathbb{R}$ such that $|f|^p$ is integrable, and $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{C})$ is the set of $\mathscr{A}$-measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is integrable.

Definition 3.14 ($\mathscr{L}^\infty$)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. We define $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$ to be the set of all[1] bounded real-valued $\mathscr{A}$-measurable functions, and $\mathscr{L}^p(X, \mathscr{A}, \mu, \mathbb{C})$ as the set of all bounded complex-valued $\mathscr{A}$-measurable functions.

Remark. Some authors[2] define $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ as the set of all *essentially bounded* $\mathscr{A}$-measurable functions on X. A function $f : X \to \mathbb{C}$ is *essentially bounded* if there exists $M > 0$ such that $\{x \in X : |f(x)| > M\}$ is locally μ-null. For most purposes, it does not matter which definition of $\mathscr{L}^\infty$ one uses. However the study of liftings is convenient with **Definition 3.14**.

Definition 3.15 (Locally μ-null)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. A subset $N \subseteq X$ is said to be *locally μ-null* if for each $A \in \mathscr{A}$ that satisfies $\mu(A) < \infty$ the set $A \cap N$ is μ-null. A property is said to hold *locally almost everywhere* if the set on which the property doesn't hold is locally μ-null.

Definition 3.16 (Seminorm on $\mathscr{L}^p$)**.** In the case of $p \in [1, \infty)$ we define a seminorm $\|\cdot\|_p : \mathscr{L}^p(X, \mathscr{A}, \mu) \to \mathbb{R}$ by

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

In the case $p = \infty$ we define a seminorm $\|\cdot\|_\infty : \mathscr{L}^\infty(X, \mathscr{A}, \mu) \to \mathbb{R}$ by

$$\|f\|_\infty = \inf \big\{ M : \{x \in X : |f(x)| > M\} \text{ is locally } \mu\text{-null} \big\}.$$

Definition 3.17 ($\mathscr{N}^p$)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\mathscr{N}^p(X, \mathscr{A}, \mu)$ be the subset of $\mathscr{L}^p(X, \mathscr{A}, \mu)$ which consists of the functions $f \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ such that $\|f\|_p = 0$. That is, if $p \in [1, \infty)$, then $\mathscr{N}^p(X, \mathscr{A}, \mu)$ is the set of functions in $\mathscr{L}^p(X, \mathscr{A}, \mu)$ which vanish almost everywhere, and if $p = \infty$ then $\mathscr{N}^\infty(X, \mathscr{A}, \mu)$ is the set of functions in $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ which vanish locally almost everywhere.

Definition 3.18 (L^p)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. We define $L^p(X, \mathscr{A}, \mu)$ to be the quotient group $\mathscr{L}^p(X, \mathscr{A}, \mu)/\mathscr{N}^p(X, \mathscr{A}, \mu)$. That is $L^p(X, \mathscr{A}, \mu)$ is the collection of cosets of

[1] I think it's supposed to be almost everywhere bounded functions, otherwise exercise 3.3.7 fails with this definition (however not with the alternative definition).

[2] Notably, the first edition of Cohn's *Measure Theory* uses this definition.

$\mathscr{N}^p(X, \mathscr{A}, \mu)$ in $\mathscr{L}^p(X, \mathscr{A}, \mu)$; these cosets are by definition the equivalence classes induced by the equivalence relation $\sim$, where $f \sim g$ holds if and only if $f - g \in \mathscr{N}^p(X, \mathscr{A}, \mu)$. Then if $p \in [1, \infty)$, $f \sim g \iff f = g$ almost everywhere.

Definition 3.19 (Norm on L^p)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. For each $f \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ let $\langle f \rangle$ be the coset of $\mathscr{N}^p(X, \mathscr{A}, \mu)$ in $\mathscr{L}^p(X, \mathscr{A}, \mu)$ to which f belongs. Then $L^p(X, \mathscr{A}, \mu)$ is a vector space and we can define a norm $\|\cdot\|_p : L^p(X, \mathscr{A}, \mu) \to \mathbb{R}$ by $\|\langle f \rangle\|_p = \|f\|_p$, where on the right hand side $\|\cdot\|_p : \mathscr{L}^p(X, \mathscr{A}, \mu) \to \mathbb{R}$ is given in **Definition 3.16**.

Definition 3.20 (Convergence in pth mean)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, let $p \in [1, \infty)$, and let $f, f_1, f_2, \dots \in \mathscr{L}^p(X, \mathscr{A}, \mu)$. Then $\{f_n\}$ converges to f in *pth mean*, or in *L^p norm*, if $\lim_n \|f_n - f\|_p = 0$.

3.3 Dual Spaces

Definition 3.21 (Linear operator)**.** Let V_1, V_2 be normed vector spaces over $\mathbb{C}$ (or over $\mathbb{R}$), then a function $T : V_1 \to V_2$ is a *linear operator* or *linear transformation* if $T(\alpha v) = \alpha T(v)$ and $T(u + v) = T(u) + T(v)$ hold for all $\alpha \in \mathbb{C}$ (or $\mathbb{R}$) and all $u, v \in V_1$.

Definition 3.22 (Bounded linear operator)**.** Let V_1, V_2 be normed vector spaces, and let $T : V_1 \to V_2$ be linear. Then a nonnegative number A such that $\|T(v)\| \leq A \|v\|$ holds for every $v \in V_1$ is called a *bound* for T, and the operator T is called *bounded* if there is a bound for it.

Definition 3.23 (Norm of linear operator)**.** Let $T : V_1 \to V_2$ be a bounded linear operator, we define the *norm* of T by

$$\|T\| = \inf\{A : A \text{ is a bound for } T\}.$$

Then $\|\cdot\|$ is a norm on the vector space of bounded linear operators from V_1 to V_2.

Definition 3.24 (Isometry)**.** Let $T : V_1 \to V_2$ be a linear operator between normed linear spaces. Then T is called and *isometry* if $\|T(v)\| = \|v\|$ for every $v \in V_1$.

Definition 3.25 (Isometric isomorphism)**.** Let $T : V_1 \to V_2$ be a linear operator between normed linear spaces. Then T is an *isometric isomorphism* if T is an isometry and is surjective. Because all isometries are injective, T is then bijective.

Definition 3.26 (Linear functional)**.** Let V be a normed linear space. A *linear functional* on V is a linear operator on V whose values lie in $\mathbb{C}$, if V is a vector space over $\mathbb{C}$, or in $\mathbb{R}$, if V is a vector space over $\mathbb{R}$.

Definition 3.27 (Dual space)**.** Let V be a normed linear space. The set of all bounded, and hence continuous, linear functionals on V then form a vector space. This vector space is called the *dual space* (or *conjugate space*) of V, and is denoted by V^*. Note that the function $\|\cdot\|\ V^* \to \mathbb{R}$ which assigns to each functional in V^* its norm, is in fact a norm on the vector space V^*.

4 Signed and Complex measures

Definition 4.1 (Signed measure)**.** Let $(X, \mathscr{A})$ be a measurable space. A function $\mu : \mathscr{A} \to [-\infty, \infty]$ is called a *signed measure* if it is countably additive and satisfies $\mu(\varnothing) = 0$.

Definition 4.2 (Positive & negative sets)**.** Let μ be a signed measure on a measurable space $(X, \mathscr{A})$. A set $A \in \mathscr{A}$ is a *positive set* if every $B \in \mathscr{A}$ such that $B \subseteq A$ satisfies $\mu(B) \geq 0$. Likewise, a set $A \in \mathscr{A}$ is a *negative set* if every $B \in \mathscr{A}$ such that $B \subseteq A$ satisfies $\mu(B) \leq 0$.

Definition 4.3 (Hahn decomposition)**.** A *Hahn decomposition* of a signed measure μ on the measurable space $(X, \mathscr{A})$ is a pair (P, N) of disjoint subsets in $\mathscr{A}$ such that $X = P \cup N$, and P is a positive set and N is a negative set. Note that there may be several Hahn decomposition of the signed measure μ.

Definition 4.4 (Complex measure)**.** Let $(X, \mathscr{A})$ be a measurable space. A *complex measure* is a function $\mu : \mathscr{A} \to \mathbb{C}$ that satisfies $\mu(\varnothing) = 0$ and is countably additive. A complex measure μ can be written as $\mu = \mu' + i\mu''$ where μ' and μ'' are finite signed measures.

Definition 4.5 (Jordan decomposition)**.** Let μ be a signed measure on the measurable space $(X, \mathscr{A})$, and let (P, N) be a Hahn decomposition of μ. Let $\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$, then μ^+, μ^- are measures on $(X, \mathscr{A})$ and $\mu = \mu^+ - \mu^-$. The measures μ^+ and μ^- are called the *positive part* and *negative part* of μ, respectively. The representation $\mu = \mu^+ - \mu^-$ is called the *Jordan decomposition* of the signed measure μ. If μ is a complex measure on $(X, \mathscr{A})$ then the representation $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ is called the *Jordan decomposition* of μ, if $\mu' = \mu_1 - \mu_2$ and $\mu'' = \mu_3 - \mu_4$ are the Jordan decompositions of the real and imaginary parts of μ.

Definition 4.6 (Variation)**.** If μ is a signed measure on the measurable space $(X, \mathscr{A})$, then the *variation* of μ is defined to be $|\mu| = \mu^+ + \mu^-$, and the *total variation* of μ is defined to be $\|\mu\| = |\mu|(X)$. If μ is a complex measure on $(X, \mathscr{A})$, then the *variation* of μ is defined by

$$|\mu|(A) = \sup \left\{ \sum_{j=1}^{n} |\mu(A_j)| : \{A_j\}_{j=1}^{n} \text{ are finite disjoint sequences in } \mathscr{A} \text{ such that } A = \bigcup_{j=1}^{n} A_j \right\}.$$

The total variation of μ is defined to be $\|\mu\| = |\mu|(X)$.

Definition 4.7. Let $(X, \mathscr{A})$ be a measurable space. Define $M(X, \mathscr{A}, \mathbb{R})$ as the set of all finite signed measures on $(X, \mathscr{A})$, and $M(X, \mathscr{A}, \mathbb{C})$ as the set of all complex measures on $(X, \mathscr{A})$. It is easy to see that $M(X, \mathscr{A}, \mathbb{R})$ and $M(X, \mathscr{A}, \mathbb{C})$ are vector spaces over $\mathbb{R}$ and $\mathbb{C}$ respectively, and that the total variation gives a norm on each of them.

Definition 4.8 (Integration with signed measure)**.** Let $X, \mathscr{A})$ be a measurable space. Denote by $B(X, \mathscr{A}, \mathbb{R})$ the vector space of bounded real-valued $\mathscr{A}$-measurable functions on X.

If μ is a finite signed measure on $(X, \mathscr{A})$, and $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ, and if $f \in B(X, \mathscr{A}, \mathbb{R})$, then the *integral of f with respect to* μ is defined as

$$\int f\, d\mu = \int f\, d\mu^+ - \int f\, d\mu^-.$$

Definition 4.9 (Integration with complex measure)**.** Let $X, \mathscr{A})$ be a measurable space. Denote by $B(X, \mathscr{A}, \mathbb{C})$ the vector space of bounded complex-valued $\mathscr{A}$-measurable functions on X. If μ is a complex measure on $(X, \mathscr{A})$, and μ_1, μ_2 are the real and imaginary parts of μ, and if $f \in B(X, \mathscr{A}, \mathbb{C})$, then the *integral of f wuth respect ti* μ is defined by

$$\int f\, d\mu = \int f\, d\mu_1 + i \int f\, d\mu_2.$$

Remark. The formula $f \mapsto \int f\, d\mu$ and $\mu \mapsto \int f\, d\mu$ define a linear functionals on $B(X, \mathscr{A})$ and $M(X, \mathscr{A})$ respectively.

Definition 4.10 (Absolute continuity)**.** Let $(X, \mathscr{A})$ be a measureable space, and let μ and ν be measures on $(X, \mathscr{A})$. We say that ν *is absolutely continuous with respect to* μ if every $A \in \mathscr{A}$ such that $\mu(A) = 0$ also satisfies $\nu(A) = 0$. This is sometimes indicated as $\nu \ll \mu$. A measure ν on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is called absolutely continuous if $\nu \ll \lambda$.

Definition 4.11 (Absolute continuity of signed or complex measure)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space. A signed or complex measure ν on $(X, \mathscr{A})$ is *absolutely continuous with respect to* μ, written $\nu \ll \mu$, if the variation $|\nu|$ is absolutely continuous with respect to μ.

Definition 4.12 (Radon-Nikodym derivative)**.** Let $(X, \mathscr{A})$ be a measurable space, let μ be a σ-finite meaure on $(X, \mathscr{A})$ and let ν be a, finite signed, complex, or σ-finite, meaure on $(X, \mathscr{A})$ such that $\nu \ll \mu$. A function g such that $\nu(A) = \int_A g\, d\mu$ hold for every $A \in \mathscr{A}$ is called a *Radon-Nikodym derivative* of ν with respect to μ, or in light of the μ-almost uniqueness of such g, *the Radon-Nikodym derivative* of ν with respect to μ. The Radon-Nikodym derivative of ν is sometimes denoted $\frac{d\nu}{d\mu}$.

Definition 4.13 (Concentrated measure)**.** Let $(X, \mathscr{A})$ be a measurable space, a measure μ is *concentrated on* $E \in \mathscr{A}$ if $\mu(E^c) = 0$. A signed or complex measure μ is said to be concentrated on E if $|\mu|(E^c) = 0$.

Definition 4.14 (Singularity)**.** Let $(X, \mathscr{A})$ be a measurable space, let μ and ν be positive, signed, or complex measures on $(X, \mathscr{A})$. Then μ and ν are called *mutually singular* if there exists $E \in \mathscr{A}$ such that μ is concentrated on E and ν is concentrated on E^c. That two measures are mutually singular is sometimes denoted $\mu \perp \nu$. Sometimes the statement μ and ν are mutually singular is said, μ and ν are singular, μ is singular with respect to ν, or that ν is singular with respect to μ. A positive, signed, or complex measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is simply said to be *singular* if it is singular with respect to the d-dimensional Lebesgue measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.

Definition 4.15 (Lebesgue decomposition)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space, and let ν be a finite signed, complex, or σ-finite positive measure on $(X, \mathscr{A})$. There are unique finite signed, complex, or σ-finite measures ν_a and ν_s on $(X, \mathscr{A})$ that satisfy

(a) $\nu_a \ll \mu$,

(b) $\nu_s \perp \mu$, and

(c) $\nu = \nu_a + \nu_s$.

The decomposition $\nu = \nu_a + \nu_s$ is called *the Lebesgue decomposition of* ν.

5 Product Measures

Definition 5.1 (Product of σ-algebras)**.** Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A subset of $X \times Y$ is called a *rectangle with measurable sides* if it has the form $A \times B$ for some $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The σ-algebra on $X \times Y$ generated by collection of rectangles with measurable sides is called the *product* of $\mathscr{A}$ and $\mathscr{B}$, and is denoted by $\mathscr{A} \times \mathscr{B}$.

Definition 5.2 (Product measure)**.** Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces. The unique measure $\mu \times \nu$ on $\mathscr{A} \times \mathscr{B}$ which satisfies $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$, for every $A \in \mathscr{A}, B \in \mathscr{B}$, is called the *product* of μ and ν.

10 Probability

Definition 10.1 (Probability space)**.** A *probability space* is a measure space $(\Omega, \mathscr{A}, P)$ such that $P(\Omega) = 1$. The elements of Ω are called the *elementary outcomes* or the *sample points* of our experiment, and the members of $\mathscr{A}$ are called *events*. If $A \in \mathscr{A}$, then $P(A)$ is the *probability* of the event A.

Definition 10.2 (Random variable)**.** A *real-valued random variable* on a probability space $(\Omega, \mathscr{A}, P)$ is an $\mathscr{A}$-measurable function from Ω to $\mathbb{R}$. Such a variable represents a numerical observation or measurement whose value depends on the outcome of the random experiment represented by $(\Omega, \mathscr{A}, P)$. More generally, a *random variable* with values in a measurable space $(S, \mathscr{B})$ is a measurable function from $(\Omega, \mathscr{A}, P)$ to $(S, \mathscr{B})$.

Definition 10.3 (Distribution)**.** Let X be a random variable with values in $(S, \mathscr{B})$. The the *distribution* of X is the measure PX^{-1} (see **Definition 2.11**) defined on $(S, \mathscr{B})$ by $(PX^{-1})(A) = P(X^{-1}(A))$. We will often write P_X for the distribution of a random variable X. If $X_1, \ldots, X_d$ are $(S, \mathscr{B})$-valued random variables on $(\Omega, \mathscr{A}, P)$, then the formula $X(\omega) = (X_1(\omega, \ldots, X_d(\omega))$ defines an S^d-valued random variable X; the distribution of X is called the *joint distribution* of $X_1, \ldots, X_d$.

Definition 10.4 (Expected value)**.** If a real-valued random variable on the probability space $(\Omega, \mathscr{A}, P)$ is integrable, then the *expected value* of X is defined $E(X) = \int X \, dP$. The expected value of X is often denoted μ_X.

Definition 10.5 (Variance)**.** If X is a real-valued random variable, then the *variance* of X is the expected value of the random variable $(X - E(X))^2$, often denoted $\mathrm{Var}(X)$ or σ_X^2. The numerical value $\sqrt{\sigma_X^2} = \sigma_X$ is called the *standard deviation* of X.

Definition 10.6. If X is $\mathbb{R}^d$ valued and $P_X \ll \lambda$, then the Radon-Nikodym derivative of P_X f_X, is called the *density function* of X.

Definition 10.7 (Independence)**.** Let $(\Omega, \mathscr{A}, P)$ be a probability space, and led $\{A_i\}_{i \in I}$ be an indexed family of events in $\mathscr{A}$. The events A_i are called *independent* if for each finite subset I_0 of I we have $P(\cap_{i \in I_0} A_i) = \prod_{i \in I_0} P(A_i)$. Let $\{X_i\}_{i \in I}$ be an indexed family of random variables defined on $(\Omega, \mathscr{A}, P)$ and with values in the measurable space $(S, \mathscr{B})$. The random variables X_i are aclled *independent* if for each choice of sets $A_i \in \mathscr{B}$, $i \in I$, the events $X_i^{-1}(A_i)$ are independent. Finally if $\{\mathscr{A}\}_{i \in I}$ is an indexed family of sub-σ-algebras of $\mathscr{A}$, then the σ-algebras $\mathscr{A}_i$ are independent if for each choice $A_i \in \mathscr{A}_i$ the events A_i are independent.